

COHOMOLOGICAL DIMENSION, CONNECTIVITY, AND LUSTERNIK–SCHNIRELMANN CATEGORY

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ABSTRACT. Dranishnikov [D2] proved that

$$\operatorname{cat} X \leq \operatorname{cd}(\pi_1(X)) + \left\lceil \frac{\operatorname{hd}(X) - 1}{2} \right\rceil.$$

where $\operatorname{cd}(\pi)$ denotes the cohomological dimension of a group π and $\operatorname{hd}(X)$ denotes the homotopy dimension of X . Furthermore, there is a well-known inequality of Grossman, [G]:

$$\operatorname{cat} X \leq \left\lceil \frac{\operatorname{hd}(X)}{k+1} \right\rceil \text{ if } \pi_i(X) = 0 \text{ for } i \leq k.$$

We make a synthesis and generalization of both of these results, by demonstrating the main result:

$$\operatorname{cat} X \leq \operatorname{cd}(\pi_1(X)) + \left\lceil \frac{\operatorname{hd}(X) - 1}{k+1} \right\rceil \text{ if } \pi_i(X) = 0 \text{ for } i = 2, \dots, k.$$

The proof of the main theorem uses the Oprea–Strom inequality $\operatorname{cat} X \leq \operatorname{hd}(B\pi_1(X)) + \operatorname{cat}^1 X$, [OS] where cat^1 is the Clapp–Puppe $\operatorname{cat} \mathcal{A}$ with \mathcal{A} the class of 1-dimensional CW complexes. The inequality clarified the Dranishnikov inequality.

1. INTRODUCTION

We work in the category of connected CW complexes and continuous maps. We use the sign \cong for homotopy equivalences. All covers are assumed to be open. Given a space X , $\operatorname{hd}(X)$ denotes the *homotopy dimension* of X , that is, the minimum cellular dimension of all CW complexes homotopy equivalent to X . Given a group π , $B\pi$ denotes a *classifying space* for π , and $\operatorname{cd}(\pi)$ denotes the *cohomological dimension* of π , [B]. A *classifying map* for X is a map $c = c_X : X \rightarrow B\pi_1(X)$ that induces an isomorphism of fundamental groups.

Below $\operatorname{cat} X$ denotes the *Lusternik–Schnirelmann category* of a space X , [LS, CLOT]. A well-known inequality $\operatorname{cat} X \leq \operatorname{hd} X$, [LS, F, CLOT] can be generalized as follows [G, CLOT]:

1.1. Theorem. *If $\pi_i(X) = 0$ for $i \leq k$ then $\operatorname{cat} X \leq \frac{\operatorname{hd} X}{k+1}$.*

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Concerning the case of $\pi_1(X) \neq 0$, the author conjectured that $\text{cat } X$ can be asymptotically bounded above by $\text{hd}(X)/2$, provided that $\pi_1(X)$ has finite cohomological dimension, i.e., $\text{cd}(\pi_1(X)) < \infty$. Later Dranishnikov [D2] proved the following fact:

1.2. Theorem. $\text{cat } X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\text{hd}(X) - 1}{2} \right\rceil$.

This theorem can be regarded as a confirmation of the conjecture.

Also, I suspected that there should be a synthesis of Theorem 1.1 and Theorem 1.2, so that the equation Theorem 1.2 can be improved by replacing (approximately) $\text{hd } X$ by $\text{hd } X/(k+1)$ for X k -connected. In other words, I expected to have a claim that generalizes both Theorem 1.1 and Theorem 1.2. Now I know how to make this improvement (synthesis, generalization). Let me tell you the precise statements (Theorem 1.6 and Corollary 1.7 below).

1.3. Definition (Clapp and Puppe [CP]). Given a class \mathcal{A} of CW complexes and a space X , define a subset A of X to be \mathcal{A} -categorical if the inclusion $A \rightarrow X$ factors, up to homotopy, through a space in \mathcal{A} . Follow Clapp and Puppe [CP], define the \mathcal{A} -cover of X to be the cover $\{U_0, U_1, \dots, U_m\}$ such that each U_i is \mathcal{A} -categorical. Define $\text{cat}_{\mathcal{A}} X$, the \mathcal{A} -category of X to be the minimal number k such that there exists an \mathcal{A} -categorical cover $\{U_0, U_1, \dots, U_k\}$.

For example, $\text{cat}_{\mathcal{A}}(X) = \text{cat}(X)$ if \mathcal{A} is the class of contractible spaces.

1.4. Definition. Let $\mathcal{A}(n)$ be the class of all n -dimensional CW complexes. Put $\text{cat}^n(X) := \text{cat}_{\mathcal{A}(n)}(X)$.

The following Oprea–Strom Theorem recovers and clarifies Theorem 1.2.

1.5. Theorem. *For every space X we have*

$$\text{cat } X \leq \text{hd}(B\pi_1(X)) + \text{cat}^1(X) \leq \text{hd}(B\pi_1(X)) + \left\lceil \frac{\text{hd}(X) - 1}{2} \right\rceil.$$

Proof. See Oprea and Strom [OS, Corollary 6.2]. \square

We prove the following generalization of Theorem 1.5:

1.6. Theorem (Corollary 2.5). *Let X be a CW complex and $\pi = \pi_1(X)$. Suppose that the classifying map $c : X \rightarrow B\pi$ induces an isomorphism $c_* : \pi_i(X) \rightarrow \pi_i(B\pi)$ for $i \leq k$. (In particular, $\pi_i(X) = 0$ for $i = 2, \dots, k$.) Then*

$$\text{cat } X \leq \text{hd}(B\pi) + \text{cat}^k(X) \leq \text{hd}(B\pi) + \left\lceil \frac{\text{hd}(X) - 1}{k + 1} \right\rceil.$$

1.7. Corollary. *Let X be a CW complex as in Theorem 1.6. Then*

$$\text{cat } X \leq \text{cd}(\pi) + \left\lceil \frac{\text{hd}(X) - 1}{k + 1} \right\rceil.$$

Clearly, Theorem 1.6 and Corollary 1.7 can be regarded as an above-mentioned synthesis.

2. PROOFS

First, we settle the second part of the inequality noted in Theorem 1.6.

2.1. Proposition. *Let X_k be the k -skeleton of a CW complex X . Then*

$$\text{cat}^k(X) \leq \left\lfloor \frac{\text{hd}(X)}{k + 1} \right\rfloor \leq \left\lceil \frac{\text{hd}(X) - 1}{k + 1} \right\rceil$$

Proof. For the first inequality, see [OS, Proposition 4.4]. The second inequality is obvious. \square

Now we prove the first part of 1.6. The proof is based (speculated) on [OS, Sections 5,6] that, in turn, exploits clever ideas of Dranishnikov [D1, D2].

Let X be a CW complex. Take $k > 1$, let X_k be the k -skeleton of X , and let \tilde{Z} be the universal covering of Z for $Z = X$ or $Z = X_k$. Put $\pi = \pi_1(X)$ and let $E\pi \rightarrow B\pi$ be the universal bundle for π . Note that π acts on \tilde{Z} via deck transformations of the covering $\tilde{Z} \rightarrow Z$, and we can form the Borel construction

$$(2.1) \quad p : E\pi \times_{\pi} \tilde{Z} \rightarrow B\pi.$$

It is worth noting that $E\pi$ is contractible, and so

$$(2.2) \quad E\pi \times_{\pi} \tilde{Z} \cong Z.$$

The inclusion $i = i_k : X_k \rightarrow X$ yields the commutative diagram

$$\begin{array}{ccc} E\pi \times_{\pi} \tilde{X}_k & \xrightarrow{f} & E\pi \times_{\pi} \tilde{X} \\ p_0 \downarrow & & \downarrow p_1 \\ B\pi & \xlongequal{\quad} & B\pi \end{array}$$

where $f = f_k$ is induced by i , and $p_0 = p$ if $Z = X_k$, and $p_1 = p$ if $Z = X$. Take a base point $*$ of $B\pi$ and let F_0, F_1 be the fibers of p_0, p_1 over $*$, respectively. Then f yields a map (inclusion) $j : F_0 \rightarrow F_1$ of fibers.

2.2. Proposition. *If $\pi_i(X) = 0$ for $i = 2, \dots, k$ then the inclusion $j : F_0 \rightarrow F_1$ is null-homotopic.*

Proof. It follows because F_0 is homotopy equivalent to \tilde{X}_k , while \tilde{X}_k is contractible ($\pi_i(\tilde{X}_k) = 0$ for $i \leq k$ and $\text{hd } X_k \leq k$). \square

Following [OS], define a cover $\mathcal{U} = \{U_0, U_1, \dots, U_n\}$ of $E\pi \times_\pi \tilde{X}$ to be a Γ_k -cover if each inclusion $U_m \subset E\pi \times_\pi \tilde{X}$ passes through the inclusion f , up to homotopy over $B\pi$. In this case we define $\gamma(\mathcal{U}) = n$. Now, set

$$\Gamma_k(X) = \inf\{\gamma(\mathcal{U}) \mid \mathcal{U} \text{ is a } \Gamma_k\text{-cover of } E\pi \times_\pi \tilde{X}\}.$$

2.3. Proposition. *We have $\text{cat } X = \text{cat}(E\pi \times_\pi \tilde{X}) \leq \text{hd } B(\pi) + \Gamma_k$.*

Proof. The equality $\text{cat } X = \text{cat}(E\pi \times_\pi \tilde{X})$ is explained in (2.2). The inequality follows from [OS, Prop. 5.1] because of Proposition 2.2. \square

2.4. Theorem. *For any CW complex X and every $k \geq 1$, we have $\Gamma_k(X) = \text{cat}^k(X)$*

Proof. For $k = 1$, this is [OS, Theorem 6.1]. For $k > 1$, the proof is literally the same as for $k = 1$. The only change is to replace X_1 by X_k , cat^1 by cat^k , and Γ_1 by Γ_k , in [OS, Theorem 6.1]. \square

2.5. Corollary. *Let X be a CW complex and $\pi = \pi_1(X)$. Suppose that the classifying map $c : X \rightarrow B\pi$ induces an isomorphism $c_* : \pi_i(X) \rightarrow \pi_i(B\pi)$ for $i \leq k$. Then*

$$\text{cat } X \leq \text{hd}(B\pi_1(X)) + \text{cat}^k(X) \leq \text{hd}(B\pi_1(X)) + \left\lceil \frac{\dim(X) - 1}{k + 1} \right\rceil.$$

Proof. The first inequality follows from Proposition 2.3 and Theorem 2.4, the second inequality follows from Proposition 2.1. \square

Now we prove **Corollary 1.7**. First, given a group π , recall that $\text{cd}(\pi) = \text{hd}(B\pi)$ if either $\text{cd}(\pi) \leq 3$, [EG] or $\text{cd}(\pi) = 1$, [Stal, Swan]. Furthermore, recall that $\text{cd}(\pi) = \text{cat}(B\pi)$ for all groups π , [EG, Stal, Swan]. So, for $\text{cd}(\pi) \neq 2$, Corollary 1.7 follows from Corollary 2.5 directly. (Note also that if $\text{cd}(\pi) = 2$ then either $\text{hd}(B\pi) = 2$ or $\text{hd}(B\pi) = 3$, and it is unknown question whether there exists a group π with $\text{cd}(\pi) = 2$ and $\text{hd}(B\pi) = 3$.)

Consider a space X and the classifying map $c : X \rightarrow B\pi$ where $\pi = \pi_1(X)$. Note that $c_* : \pi_1(X) \rightarrow \pi_1(B\pi)$ is an isomorphism. Given $k \in \mathbb{N}$, assume that $c_* : \pi_i(X) \rightarrow \pi_i(B\pi)$ is an isomorphism for $i \leq k$.

2.6. Lemma. *Let $f : X \rightarrow Y$ be a locally trivial bundle with an k -connected fiber F . Suppose that f admits a section. Then*

$$\text{cat } X \leq \text{cat } Y + \left\lceil \frac{\text{hd}(X) - k}{k + 1} \right\rceil.$$

Proof. See [D1, Theorem 3.7]. \square

We apply Lemma 2.6 to the Borel construction $p : E\pi \times_\pi \tilde{X} \rightarrow B\pi$ as in (2.1), with $\text{cd}(\pi) = 2$. Note that the bundle p is the classifying map for X . Furthermore, the fiber F of p is homotopy equivalent to \tilde{X} .

For $k = 1$, Corollary 1.7 is the Dranishnikov theorem Theorem 1.2. So, assume that $k > 1$. Then $\pi_2(F) = \pi_2(\tilde{X}) = 0$, since $\pi_2(\tilde{X}) = \pi_2(X) = \pi_2(B\pi)$.

We have $\text{hd}(B\pi) \leq 3$ and $\pi_i(F) = 0$ for $i = 1, 2$. So, because of the elementary obstruction theory, p has a section. Thus, because of 2.6 and since $\text{cd}(\pi) = \text{cat}(B\pi)$, we conclude that

$$\text{cat } X \leq \text{cd } \pi + \left\lceil \frac{\text{hd}(X) - 1}{k + 1} \right\rceil$$

for $\text{cd}(\pi) = 2$, and therefore for all π . This completes the proof of Corollary 2.5.

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